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HOW THE MAP-PROBLEM WAS SOLVED IN THE WAR.[1]

By KURT LAVES, University of Chicago.

Introductory Considerations which Led to the Selection of the Lambert Projection for the War Maps of Northern France.

For the pre-war maps of northern France the Bonne projection was used, with the Meridian of Paris as primary meridian. The Bonne projection (Rigobert Bonne, 1727–1795) is a modified conical projection. As an "equivalent" projection any area on the map is equal to the corresponding area on the Earth's surface. For all peaceful occupations this is an essential requirement, while for the purposes of war, in particular those of heavy artillery, the essential requirement is that the angles and distances radiating from *any point* of the territory mapped should with only inappreciable errors be obtainable from the data furnished by the map. The theory of the Bonne projection will show that the distortions in angle and distance increase with the longitude from the primary meridian. At the eastern frontier of northern France the angular errors[2] amount to 18′ and the linear errors to 1 : 379. Such errors are not negligible in artillery fire. The eventual subsequent transfer of the battlefront in an easterly direction for which the military command had to make early provision in their map-department would have necessitated the introduction of a second and perhaps of a third and fourth primary meridian under the Bonne projection, in order that these errors might be kept within specified limits. Two, and eventually three or even more, sets of maps which furnish *different* data for the territory covered by them in common are a source of confusion which may lead to disastrous results. This fact, coupled with the relatively large distortions in angle and distance at the edges of each set of maps mentioned above counted heavily against the retention of the Bonne projection. The modern fire-direction of a battery requires the determination of distance and azimuth from the rectangular coördinates of target and battery. The kilometer-grid printed on the map permits of a rapid solution of this problem. The meridians on the Bonne projection are sinusoidal curves and "convergence of meridians," *i. e.*, the angle between the meridian of a point P and that of the primary meridian, is not readily obtainable by calculation since it is a function of both latitude and longitude of P. The determination of "Bonne-North" (corresponding to Lambert-North, *i. e.*, direction of primary meridian at a point) would have proved a rather lengthy and knotty process for the average battery officer even with the help of auxiliary tables.

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[1] Read at the joint meeting of the Mathematical Association of America and of the American Mathematical Society at Chicago, December 27, 1918.

[2] Walton C. Clark, *Heavy Artillery Orientation*, p. 33; published by Coast Artillery School, Ft. Monroe, Va. $0.75.

The objections mentioned against the Bonne projection do not apply to the Lambert *conform* conical map projection for the following reasons:

1. Being conform the angles on the map are equal to those on the earth; the distortions of distances are zero along the two parallels along which the map-cone cuts the earth's surface. Between these parallels the distances on the map are smaller than those on the earth's surface, beyond them greater, the maximum linear distortion being 1:2037 for the maps of northern France which cover a belt of 5°.4 in latitude.

2. Since the distortion is not a function of the longitude from the primary meridian, the Lambert map projection allows of an unlimited extension east or west of the primary meridian. The map material is homogeneous throughout.

3. The convergence of meridians is at once obtainable from the map without the objectionable computation of the angle ψ, necessary under the Bonne projection. The longitudes on the map are μ times the corresponding longitudes on the earth where μ is a constant factor smaller than 1.

4. The principal meridian has not a determining influence on the construction of the map; it may be selected in such a way as to make the construction of the kilometer grid as convenient as circumstances will permit. The meridian of Treves served this purpose for the War maps of northern France, the tangent to the meridian at Treves being parallel to an element of the map-cone.

The Bonne Projection.

For the mapping of a country which has a considerable extent in longitude but only a moderate extent in latitude the conical projections offer the greatest advantages. The surface of a cone which touches the earth's surface along a middle parallel and whose vertex is in the prolongation of the earth's axis, will, when developed, furnish a map in which the parallel-circles of the earth are represented by concentric circles, the meridians by straight lines radiating from the vertex of the cone (center of parallel circles). While meridians and parallel-circles on the map cut each other under right angles as on the earth's surface, it is evident that the parallel circles with the exception of the middle parallel will be longer than those on the earth. The Bonne projection avoids this difficulty by sacrificing the perpendicularity of meridians and parallels except on the first meridian. On the primary meridian the degrees of latitude south and north of the middle parallel circle are measured in their true lengths. Likewise on the concentric parallel circles accurate degrees of longitude are laid down east and west of the primary meridian. The end points of equal degrees to either side are joined by smooth curves. By this device Bonne secures equivalency between areas on the earth and on the map, which for all peace purposes is obviously the feature that a map should offer. If R and L are the polar coördinates of a point on the map with the vertex of the cone for the pole and the primary meridian for axis of reference, then $R \cdot dR \cdot dL$ will be the element of area on the map. Since now $RdL = r_1 d\lambda$ and $dR = -\rho \cdot d\phi$ we have $R \cdot dR \cdot dL = -r_1 d\lambda \rho d\phi$ the corresponding area of the earth, where λ stands for longitude, ϕ for latitude, ρ for the radius of curvature of the meridian ellipse and r_1 for the radius of the parallel circle and the earth is assumed to be an ellipsoid of revolution. This shows that the Bonne projection is one of equivalence.

From the analytics of the meridian ellipse it follows that,

$$\rho = \frac{a(1 - e^2)}{\sqrt{(1 - e^2 \sin^2 \phi)^3}}, \qquad r_1 = \frac{a \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}},$$

where $2a$ is the major axis, and e the numerical eccentricity of the meridian ellipse.

The element of area will therefore be,

$$\frac{a^2(1 - e^2) \sin \chi \, d\chi \cdot d\lambda}{(1 - e^2 \cos^2 \chi)^2},$$

where $\chi = 90° - \phi$ is introduced in order that R and χ may both be increasing functions.

In order to compute the radius R of a parallel circle of the map we start from R_m, the radius of the middle parallel,

$$R_m = r_{1m} \operatorname{cosec} \phi_m = \frac{a \operatorname{ctg} \phi_m}{\sqrt{1 - e^2 \sin^2 \phi_m}}$$

and obtain

$$R = R_m - \int_{\phi_m}^{\phi} \rho d\phi = R_m + \int_{\chi_m}^{\chi} \frac{a(1 - e^2)}{\sqrt{(1 - e^2 \cos^2 \chi)^3}} d\chi.$$

The polar angle L is derived from $RdL = r_1 d\lambda$ or

$$dL = \frac{a \cos \phi \cdot d\lambda}{R \sqrt{1 - e^2 \sin^2 \phi}};$$

since R is only a function of ϕ and not of λ, we conclude,

$$L = \frac{a \cos \phi \cdot \lambda}{R \sqrt{1 - e^2 \sin^2 \phi}}, \tag{1}$$

if we count the longitudes and angles L from the primary meridian. Knowing R and L for given values of ϕ and λ, the map is easily constructed. We now turn our attention to the distortions of the map in angle and distance. We shall see that these distortions are zero for the principal meridian and the middle parallel and grow rather rapidly as the point to be represented increases in longitude and approaches the northern or southern edge of the map. To show this we consider first the law of divergence ψ of the direction of radius R of the map from the particular meridian at a point (Fig. 1). It should be remembered that L is not the longitude of the point P on the map since VP is not a meridian in the Bonne projection. In order to obtain the "convergence of meridians" with respect to the primary meridian it is necessary to find the angle ψ which the tangent to the meridian at P makes with R:

$$\operatorname{tg}\psi = \frac{R\dfrac{\partial L}{\partial\phi}\cdot d\phi}{dR};$$

since

$$R\cdot L = \frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}}\cdot\lambda,$$

we obtain

$$R\frac{\partial L}{\partial\phi}\cdot d\phi + LdR = -\frac{a(1-e^2)\lambda\sin\phi d\phi}{\sqrt{(1-e^2\sin^2\phi)^3}} = -\rho d\phi\cdot\lambda\sin\phi = dR\cdot\lambda\sin\phi.$$

The equation for $\operatorname{tg}\psi$ becomes, therefore,

$$\operatorname{tg}\psi = \lambda\sin\phi - L = \lambda\left(\sin\phi - \frac{a\cos\phi}{R\sqrt{1-e^2\sin^2\phi}}\right). \tag{2}$$

ψ will be zero for two cases, (1) $\lambda = 0$, *i. e.*, on the primary meridian, (2) on the middle parallel, since

$$\sin\phi_m = \frac{a\cos\phi_m}{R_m\sqrt{1-e^2\sin^2\phi_m}}.$$

The angle ψ for points east of the prime meridian will be in the NW. quadrant (with respect to VP) for points north of the middle parallel, and in the NE. quadrant south of this parallel.[1] If α and a are corresponding azimuths on the

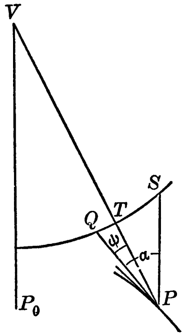

FIG. 1.

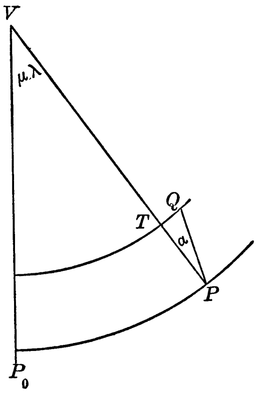

FIG. 2.

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[1] The convergence of meridians with respect to the primary meridian is the angle which the tangent to the meridian at P forms with the primary meridian. This convergence M is measured by $M = L + \psi$ or $= L + \operatorname{arc}\operatorname{tg}(\lambda\sin\varphi - L)$, or $\operatorname{tg}(M - L) = \lambda\sin\varphi - L$; for small values of ψ we have $M = \lambda\sin\varphi\cdots$. (In Major W. C. Clark's pamphlet *Heavy Artillery* (*Coast*) *Orientation*, p. 38, it is overlooked that M is a function of ψ.)

earth and the map, then $\alpha - a$ will be *the distortion in angle,* and we have

$$\operatorname{tg}(a-\psi)=\frac{QS-QT}{PT} \qquad \text{or} \qquad \operatorname{tg}(a-\psi)=\operatorname{tg}\alpha-\operatorname{tg}\psi,$$

or

$$\operatorname{tg}\alpha=\frac{\sin a}{\cos(a-\psi)\cos\psi}.$$

For the middle parallel ($\psi = 0$) we obtain $\alpha = a$.

The distortion $\alpha - a$ will be found by

$$\operatorname{tg}(\alpha-a)=\frac{\operatorname{tg}a\,[\cos a-\cos\psi\cos(a-\psi)]}{\cos(a-\psi)\cos\psi+\sin a\operatorname{tg}a}. \tag{3}$$

For $\psi = 0$ we have $\alpha = a$ as it should be. Likewise $\alpha = a$ for $tga = 0$ i. e., for the meridian.

To find the *distortion in distance* δ we have on the ellipsoid

$$\Delta\delta=\frac{a(1-e^2)\Delta\phi}{\sqrt{(1-e^2\sin^2\phi)^3}}\cdot\frac{1}{\cos\alpha},$$

and on the map

$$\Delta d=\frac{a(1-e^2)\Delta\phi}{\sqrt{1-e^2\sin^2\phi}}\cdot\frac{1}{\cos(a-\psi)}. \tag{4}$$

Hence

$$\frac{\Delta\delta-\Delta d}{\Delta d}=\frac{\cos(a-\psi)-\cos\alpha}{\cos\alpha},$$

the distortion $\Delta\delta - \Delta d$ is then likewise a function of ψ and therefore of λ and ϕ, and it increases considerably with the longitude.

The Lambert Conical Conform Projection.[1]

While in the Bonne projection the map-cone is tangential to the earth's surface at the middle parallel of the zone so that the radius of the middle map-parallel is a fundamental constant, the Lambert projection retains the properties of the cone without fixing its position from the start. Since meridians and parallel circles on the earth are orthogonal to each other a conical map projection which is to be *conform* will have to retain this orthogonal system of coördinates. Calling α the initial azimuth of a course on the surface of the earth and a the corresponding angle on the map, then for conform representation $\alpha = a$. Since $\cot\alpha = -\rho d\phi/r_1 d\lambda$, $-\rho d\phi$ being an element of the meridian arc, and $r_1 d\lambda$ that of the parallel circle, and since furthermore $\cot a = dR/RdL$, therefore $dR/RdL = \rho d\phi/r_1 d\lambda$, Fig. 2, L being here the "convergence of meridians."

We may simplify this differential equation considerably by making R only a function of ϕ and by *assuming* $dL = \mu d\lambda$, where μ is an arbitrary constant, to be determined later by special conditions.

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[1] Attention is drawn to Special Publications nos. 52 and 53 (price 25 and 10 cents respectively), of the U. S. Coast and Geodetic Survey, by Oscar S. Adams. They were issued by the Government Printing Office, Washington, in 1918, and have the following titles: *Lambert Projection Tables for the United States,* and *General Theory of the Lambert Conformal Conic Projection*—Editor.

Then we get

$$\frac{dR}{R} = \frac{-\mu\rho d\phi}{r_1} = \frac{-\mu(1-e^2)d\phi}{\cos\phi\,(1-e^2\sin^2\phi)}.$$

Integrating,

$$lg(R) = \mu lg\left[\operatorname{tg}\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{e/2} K\right],$$

K being the constant of integration. This gives

$$R = C\cdot\operatorname{tg}^{\mu}\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{\mu e/2}, \tag{5}$$

where $C = K^{\mu}$. If we put

$$\operatorname{tg}\frac{\zeta}{2} = \operatorname{tg}\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{e/2},$$

ζ will be determined to a sufficient degree of approximation (within a few tenths of a second) on setting

$$\operatorname{tg}\frac{\zeta}{2} = \operatorname{tg}\left(\frac{\pi}{4}-\frac{\phi}{2}\right).$$

Then R takes the final form

$$R = C\operatorname{tg}^{\mu}\zeta/2. \tag{6}$$

To determine the arbitrary constants μ and C, we assume that the map-cone cuts the earth's surface along the parallel circles whose latitudes are ϕ' and ϕ''. These are generally selected close to the northern and southern edges of the territory to be mapped. For the war maps it was desired to have the parallels of 52 grades (Central Switzerland) and 58 grades (Northern Holland) as extreme parallels. By taking $\phi' = 53$ grades and $\phi'' = 57$ grades the determination of the constants was effected for the war maps. To obtain μ, Lambert postulated that

$$\frac{R'}{R''} = \frac{r_1'}{r_1''};$$

this gives

$$\left(\frac{\operatorname{tg}\zeta'/2}{\operatorname{tg}\zeta''/2}\right)^{\mu} = \frac{\cos\phi'\sqrt{1-e^2\sin^2\phi''}}{\cos\phi''\sqrt{1-e^2\sin^2\phi'}},$$

or taking logarithms

$$\mu = \frac{lg\,\dfrac{\cos\phi'}{\cos\phi''}\,\dfrac{\sqrt{1-e^2\sin^2\phi''}}{\sqrt{1-e^2\sin^2\phi'}}}{lg\left(\dfrac{\operatorname{tg}\zeta'/2}{\operatorname{tg}\zeta''/2}\right)}. \tag{7}$$

In order to find C the condition is imposed that the linear distortion $k-1$ for the

Omit lines 3-6 after formula (5).

parallel of latitude ϕ' be zero; it is observed that, owing to the way μ has been found, the distortion on the parallel of ϕ'' will be zero likewise. Since

$$k = \frac{RdL}{r_1 d\lambda} = \frac{\mu R d\lambda}{r_1 d\lambda} = \mu \cdot \frac{R}{r_1},$$

we obtain

$$C = \frac{a \cos \phi'}{\mu \operatorname{tg}^{\mu} \zeta' \sqrt{1 - e^2 \sin^2 \phi'}}.$$

Since the cone is inside the earth's surface between the parallels ϕ' and ϕ'' and outside for the border area of the map, it follows that distances on the map will be smaller than corresponding distances on the earth's surface for the zone between ϕ' and ϕ'' and larger in the border-area. Computation will show that the maximum distortion amounts to 1 : 2037 or 5 : 10000; this, even for ranges of 10 km., will be a negligible quantity.

While geodetic lines on the surface of the earth are not represented by straight lines on the map but appear as slightly curved traces, the deviation for a distance of 100 km. will introduce an angular error in azimuth of less than one minute of arc.

In concluding this paper the reader's attention is once more called to Major Clark's able pamphlet which should prove a most interesting guide for students of surveying and cartography. Gretchel's *Lehrbuch der Karten-Projectionen* has been used in preparing this paper.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

32. In a discussion of the Peaucellier[1] cell by analytic methods the following equations are obtained:

$$(1)\quad (x_2 - x_1)^2 + (y_2 - y_1)^2 - b^2 = 0; \qquad (2)\quad (x_3 - x_1)^2 + (y_3 - y_1)^2 - b^2 = 0;$$

$$(3)\quad (x_2 - X)^2 + (y_2 - Y)^2 - b^2 = 0; \qquad (4)\quad (x_3 - X)^2 + (y_3 - Y)^2 - b^2 = 0;$$

$$(5)\quad x_2^2 + y_2^2 - K^2 = 0; \qquad (6)\quad x_3^2 + y_3^2 - K^2 = 0;$$

$$(7)\quad x_1^2 + y_1^2 - 2cx_1 = 0.$$

The result of eliminating x_1, y_1, x_2, y_2, x_3, y_3 gives an equation of the first degree, which establishes that the linkage will trace a straight line. There are various ways of effecting this elimination.

1. What element of the situation is left unused by the following procedure in the elimination?
   (*a*) From equations (1), (3), (5) eliminate x_2 and y_2 and obtain an equation

$$(8)\quad f_1(x_1, y_1) = 0.$$

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[1] In the accompanying figure, taken from the article on "Linkages" in the December, 1915, MONTHLY, by Mr. Leavens, the coördinates of the points of the linkage are: $O\ (0, 0)$; $C\ (c, 0)$; $P_1\ (x_1, y_1)$; $M\ (x_2, y_2)$; $M'\ (x_3, y_3)$; $P_2\ (X, Y)$.